Homework exercises Algebraic topology, hand in before class on 24-10-2018

Exercise 1. (coffee mug and donut)
If you want to be a topologist you should practice homotoping a coffee cup into a donut. In the language of CW complexes it goes like this.
Let C be the CW complex with $C_0 = \{u, d\}$ and $C_1 = C_0 \cup_{J_1\times\partial D^1} J_1 \times D^1$ with $J_1 = \{U, D, V, H\}$ and the gluing map $f^1 : J_1 \times \partial D^1 \to C_0$ defined by $f^1(U, \pm1) = u$, $f^1(D, \pm1) = d$, $f^1(V, -1) = f^1(H, -1) = d$, $f^1(V, 1) = f^1(H, 1) = u$. Finally $C = C_2 = C_1 \cup_{J_2\times\partial D^2} J_2 \times D^2$ with $J_2 = \{L, M\}$ gluing map $f^2 : J_2\times\partial D^2 \to C_1$ defined as follows. Identify $\partial D^2 = S^1$ with the quotient space $[-1, 1]/\sim$ where $-1 \sim 1$ and corresponding quotient map $\pi : [-1, 1] \to S^1$. Define $g_L : [-1, 1] \to C_1$ by $g_L(t) = p(D, t)$ where $p : C_0 \sqcup J_1 \times D^1 \to C_1$ is the quotient map. Also set $g_M : [-1, 1] \to C_1$ to be

$$g_M(t) = \begin{cases} (D, 4(t+1) - 1) & \text{if } t \in [-1, -\frac{1}{2}] \\ (V, 4(t+\frac{1}{2}) - 1) & \text{if } t \in [-\frac{1}{2}, 0] \\ (U, -4t + 1) & \text{if } t \in [0, \frac{1}{2}] \\ (V, -4(t-\frac{1}{2}) + 1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

Finally f^2 is defined by $g_L = f^2|_{\{L\}\times\partial D^2} \circ \pi$ and $g_M = f^2|_{\{M\}\times\partial D^2} \circ \pi$.

a. Write down an explicit homotopy equivalence between C and the CW complex F defined below. $F_0 = \{u\}$, $F_1 = F_0 \cup_{J_1\times\partial D^1} J_1 \times D^1$. $J_1 = \{U, H\}$ attached by function h defined as $h^1(H, \pm1) = h^1(U, \pm1) = u$. Finally $F = F_2 = F_1 \cup_{J_2\times\partial D^2} J_2 \times D^2$ with $J_2 = \{L\}$ and $h^2(L, \pi(t)) = (U, t)$.

b. Compute the Euler characteristics of both C and F.

c. Now show how F deformation retracts onto the subset consisting of the 0-cell u and the 1-cell H.

d. (Bonus) Show that the circle is homotopy equivalent to a proper donut $D^2 \times S^1$.

Exercise 2. (Barycentric cell complex)
Consider the standard n-simplex Δ^n and let $E_k \subset \Delta^n$ consist of the triangles, edges and vertices (0,1 and 2-dimensional faces) of the simplices in the k-th iterated barycentric subdivision of Δ^n. For which $n \in \mathbb{N}$ can $H_n = \bigcup_{k\in\mathbb{N}} E_k \subset \Delta^n$, with the subspace topology, be a CW complex?